Results Math (2020) 75:63
Online First

https://doi.org/10.1007/s00025-020-01189-1

Results in Mathematics

Poincaré- and Sobolev- Type Inequalities for Complex m-Hessian Equations

Per Åhag and Rafał Czyż

Abstract. By using quasi-Banach techniques as key ingredient we prove Poincaré- and Sobolev- type inequalities for m-subharmonic functions with finite (p, m)-energy. A consequence of the Sobolev type inequality is a partial confirmation of Błocki's integrability conjecture for m-subharmonic functions.

Mathematics Subject Classification. Primary 35J60, 46E35, 26D10; Secondary 32U05, 31C45.

Keywords. Complex Hessian operator, m-subharmonic function, integrability, Poincaré type inequality, Sobolev type inequality.

1. Background

In 1985, Caffarelli, Nirenberg, Spruck introduced the so called real k-Hessian operator, S_k, in bounded domains in $\mathbb{R}^n$, $n \geq 2$, $1 \leq k \leq n$ [17]. The real k-Hessian operator is a nonlinear partial differential operator acting on what is known as *k-admissible* functions (also known as *k-convex* functions). A $\mathcal{C}^2$-function u is k-admissible if the following elementary symmetric functions are non-negative

$$\sigma_l(\lambda(D^2u)) = \sum_{1\leq j_j<\cdots<j_k\leq n} \lambda_{j_1}\ldots\lambda_{j_k}, \quad \text{for} \quad l = 1, \ldots, k,$$

where $\lambda(D^2u) = (\lambda_1, \ldots, \lambda_n)$ are eigenvalues of the real Hessian matrix $D^2u = [\frac{\partial^2 u}{\partial x_j \partial x_i}]$. The real k-Hessian operator is then defined by

$$S_k(u) = \sigma_k(\lambda(D^2u)).$$

Published online: 26 March 2020 Birkhäuser

By these definitions we get that the 1-Hessian operator is the classical Laplace operator defined on 1-admissible functions that are just the subharmonic functions. Furthermore, the n-Hessian operator is the real Monge–Ampère operator defined on n-admissible functions that are the same as the convex functions. Therefore, for $k = 2, \dots, n-1$, the k-Hessian operator can be regarded as a sequence of nonlinear partial differential operators linking the classical Laplace operator to the real Monge–Ampère operator. The natural progression is then to extend the set of k-admissible functions together with the real k-Hessian operator. This was done in the famous trilogy written by Trudinger and Wang [41,43,44] (especially [43]).

For $k = 1, .., n$, the k*-Hessian integral* is formally defined as

$$I_0(u) = \int_\Omega (-u) \quad \text{and} \quad I_k(u) = \int_\Omega (-u) S_k(u).$$

When $k = 1$ we see that $I_1(u) = \int_\Omega |Du|^2$ is the Dirichlet energy integral from potential theory that goes back to the work of Gauß, Dirichlet, Riemann, among many others, while for $k = n$

$$I_n(u) = \int_\Omega (-u) \det \left(D^2 u\right)$$

is the fundamental integral in the variational theory for the real Monge–Ampère equations (see e.g. [8–11,22]). The k-Hessian integral was introduced by Chou [21]. For further information about the k-Hessian integral see e.g. [49].

Now let $0 \leq l < k \leq n$, and let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a smoothly bounded $(k-1)$-convex domain, and let u be an k-admissible function that vanishes on $\partial\Omega$. Then there exists a constant $C(l, k, n, \Omega)$ depending only on n, l, k and Ω such that

$$I_l(u)^{\frac{1}{l+1}} \leq C(l, k, n, \Omega)\, I_k(u)^{\frac{1}{k+1}}. \tag{1.1}$$

For $l = 0$ and $k = 1$, we have that inequality (1.1) is

$$\int_\Omega (-u) \leq C(0, 1, n, \Omega) \left(\int_\Omega |Du|^2\right)^{1/2},$$

and this can be interpreted as a type of the classical Poincaré inequality and therefore motivates calling (1.1) a Poincaré type inequality for k-Hessian operators. Inequality (1.1) was first proved by Trudinger and Wang [42] (for an alternative proof see [29]).

Under the same requirements on Ω and u the Sobolev type inequality that is of our interest states then that there exists a constant $C(k, n, \Omega)$ depending only on k, n and Ω such that:

(1) if $1 \leq k < \frac{n}{2}$, then

$$\|u\|_{L^q} \leq C(k, n, \Omega)\, I_k(u)^{\frac{1}{k+1}}, \quad \text{for} \quad 1 \leq q \leq \frac{n(k+1)}{n-2k};$$

(2) if $k = \frac{n}{2}$, then

$$\|u\|_{L^q} \leq C(k,n,\Omega)\, I_k(u)^{\frac{1}{k+1}}, \quad \text{for} \quad q < \infty;$$

(3) if $\frac{n}{2} < k \leq n$, then

$$\|u\|_{L^\infty} \leq C(k,n,\Omega)\, I_k(u)^{\frac{1}{k+1}}.$$

If $k = 1$, then we have

$$\|u\|_{L^q} \leq C(1,n,\Omega) \left(\int_\Omega |Du|^2 \right)^{1/2}, \quad \text{for} \quad 1 \leq q \leq \frac{2n}{n-2},$$

and for $k = n$,

$$\|u\|_{L^\infty} \leq C(n,n,\Omega) \left(\int_\Omega (-u) \det\left(D^2 u\right) \right)^{\frac{1}{n+1}}.$$

The Sobolev type inequalities (1)–(3) for n-admissible functions was first proved by Chou [21], while for the general case they were proved by Wang [48] (see also [40]).

Now to the complex setting. Let $n \geq 2$ and $1 \leq m \leq n$. Mimicking the real case above we say that a $\mathcal{C}^2$-function u defined in a bounded domain in $\mathbb{C}^n$ is *m-subharmonic* or *m-admissible* if the elementary symmetric functions are positive $\sigma_l(\lambda(u)) \geq 0$ for $l = 1, \ldots, m$, where this time $\lambda(u) = (\lambda_1, \ldots, \lambda_n)$ are eigenvalues of the complex Hessian matrix $D^2_{\mathbb{C}} u = [\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}]$. The complex m-Hessian operator on a $\mathcal{C}^2$-function u is then defined by

$$\mathrm{H}_m(u) = \sigma_m(\lambda(D^2_{\mathbb{C}} u)).$$

In the complex case we get that the complex 1-Hessian operator is the classical Laplace operator defined on 1-subharmonic functions that are just the subharmonic functions, while the complex n-Hessian operator is the complex Monge–Ampère operator defined on n-subharmonic functions that is the plurisubharmonic functions. An early encounter of the complex m-Hessian operator is the work of Vinacua [45] from 1986. That work was later published in article form in [46]. The extension of m-subharmonic functions and the complex m-Hessian operator to non-smooth admissible functions was done by Błocki in 2005 [16]. There he also introduced pluripotential methods to the theory of complex Hessian operators. Standard notations and terminology in the real and complex case differ in part, and so instead of I_k above, we shall use the following notation in the complex case: For $p > 0$, $p \in \mathbb{R}$, and $m = 1, \ldots, n$, let

$$e_{0,m}(u) = \int_\Omega \mathrm{H}_m(u) \quad \text{and} \quad e_{p,m}(u) = \int_\Omega (-u)^p\, \mathrm{H}_m(u),$$

and we call $e_{p,m}(u)$ for the *(p,m)-energy of u*. Thus, for $k = 1$ we have that $e_{1,1}(u) = I_1(u)$, but notice the difference in the definition of $e_{0,1}(u)$ compared to $I_0(u)$. For the early work on the theory of variation for the complex n-Hessian operator see e.g. [12,13,20,26,27,31].

To be able to prove the Poincaré- and Sobolev- type inequalities for m-subharmonic functions we need classes of m-subharmonic functions that, in a general sense, vanishes on the boundary and additionally they should have finite (p,m)-energy. Denote these classes with $\mathcal{E}_{p,m}(\Omega)$ (see Sect. 2 for details).

Our Poincaré type inequality for the complex m-Hessian operator is:

Theorem 4.2. *Let $n\geq 2$, $1\leq l<k\leq n$, and $p\geq 0$. Assume that Ω is a bounded B_k-regular domain in $\mathbb{C}^n$. Then there exits a constant $C(p,l,k,n,\Omega)>0$, depending only on p, l, k, n, and Ω, such that for any $u\in\mathcal{E}_{p,k}(\Omega)$ we have*

$$e_{p,l}(u)^{\frac{1}{p+l}}\leq C(p,l,k,n,\Omega)e_{p,k}(u)^{\frac{1}{p+k}}. \tag{1.2}$$

If $p=0$, $l=1$, and $k=n$, then (1.2) becomes

$$\begin{aligned}\int_\Omega \Delta u &\leq C(0,1,n,n,\Omega)\left(\int_\Omega H_n(u)\right)^{\frac{1}{n}} = C(0,1,n,n\,\Omega)\left(\int_\Omega \det D^2_{\mathbb{C}}u\right)^{\frac{1}{n}}\\ &= C(0,1,n,n,\Omega)\left(\int_\Omega (dd^cu)^n\right)^{\frac{1}{n}},\end{aligned}$$

where $(dd^cu)^n$ is the standard notation for the complex Monge–Ampère operator in pluripotential theory. Furthermore, if $p=1$, $l=1$, and $k=n$, then we have that

$$\int_\Omega |Du|^2\leq C(1,1,n,n,\Omega)^2\left(\int_\Omega (-u)(dd^cu)^n\right)^{\frac{2}{n}}.$$

When Ω is assumed to have the stronger convexity property known as strongly k-pseudoconvexity, and $p=1$, then inequality (1.2) was proved by Hou [29]. In Theorem 4.3 we find the optimal constant in (1.2) for the cases $p=0$, and $p=1$, in the case of the unit ball $\Omega=\mathbb{B}$.

Our Sobolev type inequality for complex m-Hessian equations is:

Theorem 5.4. *Let $n\geq 2$, $1\leq m\leq n$, and $p\geq 0$. Assume that Ω be a bounded m-hyperconvex domain in $\mathbb{C}^n$. There exists a constant $C(p,q,m,n,\Omega)>0$, depending only on p, q, m, n, and Ω such that for any function $u\in\mathcal{E}_{p,m}(\Omega)$, and for $0<q<\frac{(m+p)n}{n-m}$, we have*

$$\|u\|_{L^q}\leq C(p,q,m,n,\Omega)e_{p,m}(u)^{\frac{1}{m+p}}. \tag{1.3}$$

For $p=0$, we have for $m=1$ and $m=n$, respectively,

$$\|u\|_{L^q}\leq C(0,q,1,n,\Omega)\int_\Omega \Delta u \qquad \text{for} \qquad 0<q<\frac{n}{n-1},$$

and

$$\|u\|_{L^q}\leq C(0,q,n,n,\Omega)\left(\int_\Omega (dd^cu)^n\right)^{\frac{1}{n}} \qquad \text{for } q>0.$$

Furthermore, for $p = 1$, we have for $m = 1$ and $m = n$, respectively,

$$\|u\|_{L^q} \le C(0, q, 1, n, \Omega) \int_\Omega |Du|^2 \qquad \text{for} \qquad 0 < q < \frac{2n}{n-1},$$

and

$$\|u\|_{L^q} \le C(0, q, n, n, \Omega) \left(\int_\Omega (-u)(dd^c u)^n \right)^{\frac{1}{n}} \qquad \text{for } q > 0.$$

For the complex n-Hessian operator with $p = 1$, inequality (1.3) was proved by Berman and Berndtsson in [15] (see also [28]). Later their result was generalized by the authors to the case when p is any positive number, and when Ω is a n-hyperconvex domain in $\mathbb{C}^n$, or a compact Kähler manifold [3]). The case when Ω is assumed to have the stronger convexity assumption of strongly k-pseudoconvexity, and $p = 1$, then inequality (1.3) was proved by Zhou [50].

After proving Theorem 5.4 we give examples that shows that the following inequalities are not in general possible:

$$e_{p,m}(u)^{\frac{1}{m+p}} \le C\|u\|_{L^q} \text{ (Example 5.5)}$$
$$\|u\|_{L^\infty} \le C e_{p,m}(u)^{\frac{1}{m+p}} \text{ (Example 5.6)}$$
$$e_{p,m}(u)^{\frac{1}{n+p}} \le C\|u\|_{L^\infty} \text{ (Example 5.7)}.$$

It is well known that all n-subharmonic functions are locally in L^p for any $p > 0$. In general, this fact is no longer valid for m-subharmonic functions. Błocki proved that if u is m-subharmonic function, then $u \in L^p_{loc}$ for $p < \frac{n}{n-m}$. Motivated by the real case he then conjectured that any m-subharmonic function is in L^p_{loc} for $p < \frac{nm}{n-m}$ [16]). Later, Dinew and Kołodziej partially confirmed this conjecture under the extra assumption that the m-subharmonic functions [24]). For the relation of this conjecture with the so called integrability exponent, and the Lelong number, of m-subharmonic functions see [14]. As an immediate consequence of our Theorem 5.4 is that we get that Błocki's conjecture is true for functions in the Cegrell class $\mathcal{E}_m(\Omega)$ (Corollary 5.8). The inequalities under investigation are very helpful in solving the Dirichlet problem for the complex Hessian type equation, and the solution of those equations can be used for the construction of certain metrics on compact Kähler and Hermitian manifolds (see e.g. [15, 28]). Furthermore, the optimal constant in these inequalities are connected to the isoperimetric inequality and therefore classically to symmetrization of functions (see e.g. [39]).
Both our proofs of Theorem 4.2, and Theorem 5.4, uses the theory of quasi-Banach spaces (Theorem 3.2).

2. Preliminaries

Here we give some necessary background. We start with the definition of m-subharmonic functions and the m-Hessian operator. Let $\Omega \subset \mathbb{C}^n$, $n \ge 2$, be a bounded domain, $1 \leq m \leq n$, and define $\mathbb{C}_{(1,1)}$ to be the set of $(1,1)$-forms with constant coefficients. With this set

$$\Gamma_m = \left\{\alpha \in \mathbb{C}_{(1,1)} : \alpha \wedge \beta^{n-1} \geq 0, \ldots, \alpha^m \wedge \beta^{n-m} \geq 0\right\} ,$$

where $\beta = dd^c|z|^2$ is the canonical Kähler form in $\mathbb{C}^n$.

Definition 2.1. Let $n \geq 2$, and $1 \leq m \leq n$. Assume that $\Omega \subset \mathbb{C}^n$ is a bounded domain, and let u be a subharmonic function defined on Ω. Then we say that u is *m-subharmonic* if the following inequality holds

$$dd^c u \wedge \alpha_1 \wedge \cdots \wedge \alpha_{m-1} \wedge \beta^{n-m} \geq 0\,,$$

in the sense of currents for all $\alpha_1, \ldots, \alpha_{m-1} \in \Gamma_m$. With $\mathcal{SH}_m(\Omega)$ we denote the set of all m-subharmonic functions defined on Ω.

Let σ_k be k-elementary symmetric polynomial of n-variable, i.e.,

$$\sigma_k(x_1, \ldots, x_n) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} x_{j_1} \cdots x_{j_k}\,.$$

It can be proved that a function $u \in \mathcal{C}^2(\Omega)$ is m-subharmonic if, and only if,

$$\sigma_k(u(z)) = \sigma_k(\lambda_1(z), \ldots, \lambda_n(z)) \geq 0,$$

for all $k = 1, \ldots, m$, and all $z \in \Omega$. Here, $\lambda_1(z), \ldots, \lambda_n(z)$ are the eigenvalues of the complex Hessian matrix $\left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z)\right]$. For $\mathcal{C}^2$ smooth m-subharmonic function u, the *complex m-Hessian operator* is defined by

$$\mathrm{H}_m(u) = (dd^c u)^m \wedge (dd^c|z|^2)^{n-m} = 4^n m!(n-m)!\sigma_m(u(z))dV_{2n},$$

where dV_{2n} is the Lebesgue measure in $\mathbb{C}^n$.

To be able to have sufficiently many m-subharmonic functions that vanishes in some sense on the boundary we need some suitable convexity condition on our underlying domain. In this paper we need m-hyperconvexity (Definition 2.2), and B_m-regularity (Definition 2.3).

Definition 2.2. Let $n \geq 2$, and $1 \leq m \leq n$. A bounded domain in $\Omega \subset \mathbb{C}^n$ is said to be *m-hyperconvex* if it admits a non-negative and m-subharmonic exhaustion function, i.e., there exits a m-subharmonic $\varphi : \Omega \to [0, \infty)$ such that the closure of the set $\{z \in \Omega : \varphi(z) < c\}$ is compact in Ω, for every $c \in (-\infty, 0)$.

Definition 2.3. Let $n \geq 2$, and $1 \leq m \leq n$. A bounded domain in $\Omega \subset \mathbb{C}^n$ is said to be *B_m-regular* if for every $f \in \mathcal{C}(\partial\Omega)$ there exists a m-subharmonic function defined on Ω such that $u = f$ on $\partial\Omega$.

Remark. (1) n-hyperconvex domains are hyperconvex domains from pluripotential theory, while 1-hyperconvex domains are regular domains in potential theory.

(2) B_n-regular domains are B-regular domains from pluripotential theory, while B_1-regular domains are regular domains in potential theory.

(3) Every B_m-regular domain is m-hyperconvex. On the other hand, the bidisc $\mathbb{D} \times \mathbb{D}$ in $\mathbb{C}^2$ is 2-hyperconvex, but not B_2-regular while it is both 1-hyperconvex and B_1-regular.
For proofs, and further information about these convexity notions see [5].

Next, we shall recall the function classes that are of our interest. As said in the introduction we shall use the following notations:

$$e_{0,m}(u) = \int_\Omega \mathrm{H}_m(u) \quad \text{and} \quad e_{p,m}(u) = \int_\Omega (-u)^p \, \mathrm{H}_m(u),$$

We say that a m-subharmonic function φ defined on a m-hyperconvex domain Ω belongs to $\mathcal{E}_m^0(\Omega)$ if φ is bounded,

$$\lim_{z \to \xi} \varphi(z) = 0 \quad \text{for every } \xi \in \partial\Omega\,,$$

and

$$\int_\Omega \mathrm{H}_m(\varphi) < \infty\,.$$

Definition 2.4. Let $n \geq 2$, $1 \leq m \leq n$, and $p \geq 0$. Assume that Ω be a bounded m-hyperconvex domain in $\mathbb{C}^n$. We say that $u \in \mathcal{E}_{p,m}(\Omega)$, if u is a m-subharmonic function defined on Ω such that there exists a decreasing sequence, $\{\varphi_j\}$, $\varphi_j \in \mathcal{E}_m^0(\Omega)$, that converges pointwise to u on Ω, as j tends to ∞, and $\sup_j e_{p,m}(\varphi_j) < \infty$.

In [32,33], it was proved that for $u \in \mathcal{E}_{p,m}(\Omega)$ the complex Hessian operator, $\mathrm{H}_m(u)$, is well-defined, and

$$\mathrm{H}_m(u) = (dd^c u)^m \wedge (dd^c|z|^2)^{n-m}\,,$$

where $d = \partial + \bar{\partial}$, and $d^c = \sqrt{-1}\big(\bar{\partial} - \partial\big)$.

Theorem 2.5 is essential when working with $\mathcal{E}_{p,m}(\Omega)$, $p > 0$.

Theorem 2.5. *Let $n \geq 2$, $1 \leq m \leq n$, and $p > 0$. Assume that Ω be a bounded m-hyperconvex domain in $\mathbb{C}^n$. For $u_0, u_1, \ldots, u_m \in \mathcal{E}_{p,m}(\Omega)$ we have*

$$\begin{aligned}&\int_\Omega (-u_0)^p dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge (dd^c|z|^2)^{n-m} \\ &\leq C\, e_p(u_0)^{p/(p+m)} e_p(u_1)^{1/(p+m)} \cdots e_p(u_m)^{1/(p+m)}\,,\end{aligned}$$

where $C \geq 1$ depends only on p, m, n and Ω.

Proof. See e.g. Lu [32,33], and Nguyễn [34]. □

Remark. If $p \neq 1$, then $C > 1$ (see [1,2,23]).

3. Quasi-Banach Spaces

In this section we introduce the necessary background of the theory of quasi-Banach spaces to be able to prove Theorem 3.2 which subsequently will be used in both the proof of Theorem 4.2 and Theorem 5.4. Let X be a real vector space. We say that $\mathcal{K}$ is a *cone* in the vector space X if it is a non-empty subset of X that satisfies:

(1) $\mathcal{K}+\mathcal{K}\subseteq\mathcal{K}$,
(2) $\alpha\mathcal{K}\subseteq\mathcal{K}$ for all $\alpha\geq 0$, and
(3) $\mathcal{K}\cap(-\mathcal{K})=\{0\}$.

It should be noted that in some texts the name *proper convex cone* is used instead. Furthermore, $\delta\mathcal{K}=\mathcal{K}-\mathcal{K}$ is vector subspace of X. Let us recall the definition of a quasi-norm and a quasi-Banach space.

Definition 3.1. A *quasi-norm* $\|\cdot\|_0$ on a cone $\mathcal{K}$ is a mapping $\|\cdot\|_0:\mathcal{K}\to[0,\infty)$ with the following properties:

(1) $\|x\|_0=0$ if, and only if, $x=0$;
(2) $\|tx\|_0=t\|x\|_0$ for all $x\in\mathcal{K}$ and $t\geq 0$;
(3) there exists a constant $C\geq 1$ such that for all $x,y\in\mathcal{K}$ we have that
$$\|x+y\|_0\leq C(\|x\|_0+\|y\|_0)\,. \tag{3.1}$$

The constant C in (3.1) is often refereed to the modulus of concavity of the quasi-norm $\|\cdot\|$. Now one can extend $\|\cdot\|_0$ to the vector space $\delta\mathcal{K}$ by
$$\|x\|=\inf\left\{\|x_1+x_2\|_0 : x=x_1-x_2, x_1,x_2\in\mathcal{K}\right\}\,.$$

The classical Aoki-Rolowicz theorem for quasi-Banach spaces ([7,37]) states that every quasi-normed space X is q-normable for some $0<q\leq 1$. In other words, X can be endowed with an equivalent quasi-norm $|||\cdot|||$ that is q-subadditive, and therefore we can define the following metric $d(x,y)=|||x-y|||^q$ on X. The vector space X is called a *quasi-Banach space* if it is complete with respect to the metric d induced by the quasi-norm $\|\cdot\|$. Note that it follows from the definition of quasi-norm that for any $x_1,\dots,x_k\in\delta\mathcal{K}$ holds
$$\|x_1+\cdots+x_k\|\leq\sum_{j=1}^{k}C^j\|x_j\|. \tag{3.2}$$

The cone $\mathcal{K}$ in a vector space X generates a vector ordering $\succcurlyeq$ defined on $\delta\mathcal{K}$ by letting $x\succcurlyeq y$ whenever $x-y\in\mathcal{K}$.

Theorem 3.2. *Let X be a real vector space, $\mathcal{K}\subset X$ a cone, and let $\|\cdot\|_0$ be a quasi-norm on $\mathcal{K}$, such that $(\delta\mathcal{K},\|\cdot\|)$ is a quasi-Banach space, and $\mathcal{K}$ is closed in $\delta\mathcal{K}$. Assume that $\Psi:X\to[0,\infty]$ is a function that satisfies:*

(a) *Ψ is homogeneous, i.e. $\Psi(tx)=t\Psi(x)$, $t\geq 0$, $x\in\mathcal{K}$;*
(b) *Ψ is increasing, i.e. if $x\succcurlyeq y$, then $\Psi(x)\geq\Psi(y)$.*

The following conditions are then equivalent:

(1) *there exists a constant $B > 0$ such that for all $x \in \mathcal{K}$ holds*
$$\Psi(x) \leq B\|x\|_0;$$

(2) *Ψ is finite on $\mathcal{K}$.*

Proof. The implication (1)⇒(2) is clear. To prove the opposite implication (2)⇒(1) we shall argue by contradiction. Assume that there does not exists any constant B as above. Therefore, by using homogeneity of Ψ we can assume that there exists a sequence $x_j \in \mathcal{K}$ such that

$$\|x_j\|_0 = 1 \qquad \text{and} \qquad \Psi(x_j) > j(2C)^j, \tag{3.3}$$

where C is the modulus of concavity of the quasi-norm $\|\cdot\|_0$. Let us define

$$y_k = \sum_{j=1}^{k}(2C)^{-j}x_j.$$

We shall prove that $\{y_k\}$ is a Cauchy sequence. By (3.2) we have that for $k > l$

$$\|y_k - y_l\|_0 = \left\|\sum_{j=l+1}^{k}(2C)^{-j}x_j\right\|_0 \leq \sum_{j=l+1}^{k} C^j(2C)^{-j}\|x_j\|_0 \leq \sum_{j=l+1}^{k} 2^{-j} < 2^{-l}.$$

Therefore, there exists $y \in \delta\mathcal{K}$ such that $y_k \to y$, as $k \to \infty$. But since the cone $\mathcal{K}$ is closed we get that $y \in \mathcal{K}$.

On the other hand, by the same argument as above we get that for any $m \in \mathbb{N}$ we have

$$y = \sum_{j=1}^{\infty}(2C)^{-j}x_j \succcurlyeq (2C)^{-m}x_m,$$

and therefore by (3.3) and monotonicity of Ψ

$$\Psi(y) = \Psi\left(\sum_{j=1}^{\infty}(2C)^{-j}x_j\right) \geq \Psi\left((2C)^{-m}x_m\right) = (2C)^{-m}\Psi\left(x_m\right) > m.$$

This is impossible by our assumption. □

Remark. Note that condition b) in Theorem 3.2 can be replaced by upper semicontinuity of Ψ.

We shall give some examples of Theorem 3.2. Example 3.3, and Example 3.4, shall be used in the proofs of Theorem 4.2, and Theorem 5.4.

Example 3.3. Assume that Ω is a bounded m-hyperconvex domain in $\mathbb{C}^n$. Let $X = L^1_{loc}(\Omega)$, $\mathcal{K} = \mathcal{E}_{p,k}(\Omega)$, and for $u \in \mathcal{K}$ let

$$\|u\|_0 = e_{p,k}(u)^{\frac{1}{p+k}}.$$

Then for any $v \in \delta\mathcal{E}_{p,k}(\Omega)$ define

$$\|u\| = \inf_{\substack{u_1 - u_2 = u \\ u_1, u_2 \in \mathcal{E}_{p,k}}} \left(\int_\Omega (-(u_1+u_2))^p \,\mathrm{H}_k(u_1+u_2) \right)^{\frac{1}{k+p}} .$$

It was proved in [1,34] that $(\delta\mathcal{E}_{p,k}, \|\cdot\|)$ is a quasi-Banach space for $p \neq 1$, and a Banach space for $p = 1$. Furthermore, the cone $\mathcal{E}_{p,k}(\Omega)$ is closed in $\delta\mathcal{E}_{p,k}(\Omega)$.

Let μ be a positive Radon measure μ, and $p > 0$. Then we define

$$\Psi_1(u) = \left(\int_\Omega (-u)^p \, d\mu \right)^{\frac{1}{p}} .$$

The functional Ψ_1 satisfies conditions $a)$ and $b)$ in Theorem 3.2. This example will be used in our proof of the Sobolev type inequality (Theorem 5.4). In this special case Theorem 3.2 was proved by Cegrell, see [18], and Lu [32,33].

Inspired by Ψ_1, we define for $1 \leq l \leq n$ the following:

$$\Psi_2(u) = \left(\int_\Omega (-u)^p \,\mathrm{H}_l(u) \right)^{\frac{1}{p+l}} .$$

This functional, Ψ_2, shall be used in the proof of the Poincaré type inequality (Theorem 4.2). □

Example 3.4. Let Ω be a bounded m-hyperconvex domain in $\mathbb{C}^n$, $n \geq 2$. Also, let $X = L^1_{loc}(\Omega)$, $\mathcal{K} = \mathcal{E}_{0,k}(\Omega)$, and for $u \in \mathcal{K}$ set

$$\|u\|_0 = e_{0,k}(u)^{\frac{1}{k}} .$$

Then for any $v \in \delta\mathcal{E}_{0,k}(\Omega)$ define

$$\|u\| = \inf_{\substack{u_1 - u_2 = u \\ u_1, u_2 \in \mathcal{E}_{0,k}}} \left(\int_\Omega \mathrm{H}_k(u_1+u_2) \right)^{\frac{1}{k}} .$$

It was proved in [19,34] that $(\delta\mathcal{E}_{0,k}(\Omega), \|\cdot\|)$ is a Banach space. Furthermore, the cone $\mathcal{E}_{0,k}(\Omega)$ is closed in $\delta\mathcal{E}_{0,k}(\Omega)$. In the proof of the Poincaré type inequality (Theorem 4.2) we shall use the following functional $(1 \leq l \leq n)$:

$$\Psi_3(u) = \left(\int_\Omega \mathrm{H}_l(u) \right)^{\frac{1}{l}} .$$

□

Example 3.5. Let $n \geq 2$, $p > 0$, and $1 \leq m \leq n$. Furthermore, assume that Ω is a bounded m-hyperconvex domain in $\mathbb{C}^n$, and let $X = \delta\mathcal{M}_{p,m}$, where

$$\mathcal{M}_{p,m} = \big\{ \mu \,:\, \mu \text{ is a non-negative Radon measure on } \Omega \text{ such that } \mathrm{H}_m(u) = \mu \text{ for some } u \in \mathcal{E}_{p,m}(\Omega) \big\} .$$

Let us here recall that the following conditions are equivalent:

(1) there exists a unique function $u \in \mathcal{E}_{p,m}(\Omega)$ such that $\mathrm{H}_m(u) = \mu$;

(2) there exists a constant $C \geq 0$ such that

$$\int_\Omega (-u)^p\, d\mu \leq C\,(e_{p,m}(u))^{\frac{p}{p+m}} \quad \text{for all } u \in \mathcal{E}_m^0(\Omega)\,;$$

(3) $\mathcal{E}_{p,m}(\Omega) \subset L^p(\mu)$.

([6,18,33,34]). For $\mu \in \delta\mathcal{M}_{p,m}$ let now $u^+, u^- \in \mathcal{E}_{p,m}(\Omega)$ be the unique m-subharmonic functions such that

$$\mathrm{H}_m(u^+) = \mu^+ = \frac{1}{2}(|\mu| + \mu), \quad \text{and} \quad \mathrm{H}_m(u^-) = \mu^- = \frac{1}{2}(|\mu| - \mu)\,.$$

Now we can define

$$|\mu|_{p,m} = \|u^+\|_{p,m}^m + \|u^-\|_{p,m}^m\,.$$

Then it was proved in [1,34] that $(\delta\mathcal{M}_{p,m}, |\cdot|_{p,m})$ is a quasi-Banach space, and for $p = 1$ a Banach space.

In this space one can consider the following functional: For $p > 0$, and a m-subharmonic function u define

$$\Psi_4(\mu) = \int_\Omega (-u)^p\, d\mu.$$

The functional, Ψ_4, satisfies conditions $a)$ and $b)$ in Theorem 3.2. In this special case Theorem 3.2 was proved in [4] in order to characterize $\mathcal{E}_{p,k}(\Omega)$ □

4. A Poincaré Type Inequality in B_k-Regular Domains

The aim of this section is to prove the Poincaré type inequality in B_k-regular domains for k-subharmonic functions. First we need the following lemma.

Lemma 4.1. *Let $n \geq 2$, $1 \leq l < k \leq n$, and $p \geq 0$. Furthermore, assume that Ω is a bounded B_k-regular domain in $\mathbb{C}^n$. Then $\mathcal{E}_{p,k}(\Omega) \subset \mathcal{E}_{p,l}(\Omega)$.*

Proof. Let $u \in \mathcal{E}_k^0(\Omega)$. Since Ω is B_k-regular we know that there exists a negative, smooth, k-subharmonic function $\varphi \in \mathcal{E}_k^0(\Omega)$ such that $(\varphi(z) - |z|^2) \in \mathcal{SH}_k(\Omega)$. Then define

$$\mu := (dd^c u)^l \wedge (dd^c|z|^2)^{n-l}.$$

Then we have

$$\begin{aligned}\mu &= (dd^c u)^l \wedge (dd^c|z|^2)^{n-l} \leq (dd^c u)^l \wedge \left(dd^c\left((\varphi - |z|^2) + |z|^2\right)\right)^{k-l} \wedge (dd^c|z|^2)^{n-k}\\ &= (dd^c u)^l \wedge (dd^c\varphi)^{k-l} \wedge (dd^c|z|^2)^{n-k} \leq (dd^c(u+\varphi))^k \wedge (dd^c|z|^2)^{n-k}. \qquad (4.1)\end{aligned}$$

Since $(u+\varphi) \in \mathcal{E}_k^0(\Omega)$ it follows that $\mathrm{H}_m(u+\varphi)$ is a finite measure, and therefore μ is also finite and $u \in \mathcal{E}_l^0(\Omega)$. Hence, $\mathcal{E}_k^0(\Omega) \subset \mathcal{E}_l^0(\Omega)$.

Case $p > 0$: Assume that $u \in \mathcal{E}_{p,k}(\Omega)$. Then by definition there exists a decreasing sequence $u_j \in \mathcal{E}_k^0(\Omega)$ such that

$$\lim_{j\to\infty} u_j = u \qquad \text{and} \qquad \sup_j e_{p,k}(u_j) < \infty.$$

Hence, $u_j \in \mathcal{E}_l^0(\Omega)$, and by Theorem 2.5 and (4.1) we get

$$\int_\Omega (-u_j)^p (dd^c u_j)^l \wedge (dd^c|z|^2)^{n-l} \leq \int_\Omega (-u_j)^p (dd^c(u_j+\varphi))^k \wedge (dd^c|z|^2)^{n-k}$$
$$\leq d(p,m,\Omega) e_{p,k}(u_j)^{\frac{p}{p+k}} e_{p,k}(u_j+\varphi)^{\frac{k}{p+k}}$$
$$\leq d(p,m,\Omega) e_{p,k}(u_j)^{\frac{p}{p+k}} \left(d(p,m,\Omega)\left(e_{p,k}(u_j)^{\frac{1}{p+k}} + e_{p,k}(\varphi)^{\frac{1}{p+k}}\right)\right)^k.$$

Hence, $\sup_j e_{p,l}(u_j) < \infty$. Thus, $u \in \mathcal{E}_{p,l}(\Omega)$.

Case $p = 0$: Assume that $u \in \mathcal{E}_{0,k}(\Omega)$. By definition there exists a decreasing sequence $u_j \in \mathcal{E}_k^0(\Omega)$ such that

$$\lim_{j\to\infty} u_j = u \qquad \text{and} \qquad \sup_j e_{0,k}(u_j) < \infty.$$

Hence, $u_j \in \mathcal{E}_l^0$ and therefore by [30] and (4.1) we get

$$\int_\Omega (dd^c u_j)^l \wedge (dd^c|z|^2)^{n-l} \leq \int_\Omega (dd^c(u_j+\varphi))^k \wedge (dd^c|z|^2)^{n-k}$$
$$\leq \left(e_{0,k}(u_j)^{\frac{1}{k}} + e_{0,k}(\varphi)^{\frac{1}{k}}\right)^k.$$

This means that $\sup_j e_{0,l}(u_j) < \infty$, so $u \in \mathcal{E}_{0,l}(\Omega)$. □

Remark. Let $\Omega = \mathbb{D}^2$ be the bidisc in $\mathbb{C}^2$. This domain is 2-hyperconvex, but not B_2-regular. Let

$$u(z_1,z_2) = \sum_{k=1}^{\infty} \max(\log|z_1|, k^{-4}\log|z_2|)$$

be defined on Ω. Then $u \in \mathcal{E}_{0,2}(\Omega)$, but it is not in $\mathcal{E}_{0,1}(\Omega)$ (see [19] for details). Next, define

$$v(z_1,z_2) = \sum_{j=1}^{\infty} \max(j^{-6}\ln|z_1|, \ln|z_2|, -1)\,.$$

By straight forward calculations we see that $v \in \mathcal{E}_{0,2}(\Omega) \cap \mathcal{E}_{1,2}(\Omega)$, but it is not in $\mathcal{E}_{0,1}(\Omega) \cup \mathcal{E}_{1,1}(\Omega)$.

Now to the proof of the Poincaré type inequality.

Theorem 4.2. *Let $n \geq 2$, $1 \leq l < k \leq n$, and $p \geq 0$. Assume that Ω is a bounded B_k-regular domain in $\mathbb{C}^n$. Then there exits a constant $C(p,l,k,n,\Omega) > 0$, depending only on p, l, k, n, and Ω, such that for any $u \in \mathcal{E}_{p,k}(\Omega)$ we have*

$$e_{p,l}(u)^{\frac{1}{p+l}} \leq C(p,l,k,n,\Omega) e_{p,k}(u)^{\frac{1}{p+k}}.$$

Proof. Using the functionals Ψ_2 and Ψ_3 (from Example 3.3 and Example 3.4) the proof follows from Theorem 3.2 and Lemma 4.1. □

Next, we shall determine the optimal constant in Theorem 4.2 for the unit ball in $\mathbb{C}^n$ in the cases $p = 0$ and $p = 1$.

Theorem 4.3. *Let $n \geq 2$, $1 \leq l < k \leq n$, and $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$. The optimal constant $C(p, l, k, n, \mathbb{B})$ in Theorem 4.2 is given by:*

(a) $C(0, l, k, n, \mathbb{B}) = (4\pi)^{\frac{n}{l} - \frac{n}{k}}$ $(p = 0)$;

(b) $C(1, l, k, n, \mathbb{B}) = \left(\frac{(4\pi)^n}{n+1}\right)^{\frac{1}{l} - \frac{1}{k}}$ $(p = 1)$.

Proof. *Case* $p = 0$: We shall start proving that there exists a constant $C > 0$ such that for any $u \in \mathcal{E}_{0,m}(\mathbb{B})$ it holds

$$\left(\int_{\mathbb{B}} (dd^c u)^l \wedge (dd^c|z|^2)^{n-l}\right)^{\frac{1}{l}} \leq C \left(\int_{\mathbb{B}} (dd^c u)^k \wedge (dd^c|z|^2)^{n-k}\right)^{\frac{1}{k}}. \tag{4.2}$$

Set $\beta = dd^c(|z|^2 - 1)$, and note that $|z|^2 - 1$ is an exhaustion function for $\mathbb{B}$. Then for any $u \in \mathcal{E}_{0,m}(\mathbb{B})$. We get by [30]

$$\begin{aligned}\int_{\mathbb{B}} (dd^c u)^l \wedge (dd^c|z|^2)^{n-l} &= \int_{\mathbb{B}} (dd^c u)^l \wedge (dd^c(|z|^2 - 1))^{k-l} \wedge (dd^c|z|^2)^{n-k} \\ &\leq \left(\int_{\mathbb{B}} (dd^c u)^k \wedge (dd^c|z|^2)^{n-k}\right)^{\frac{l}{k}} \left(\int_{\mathbb{B}} (dd^c|z|^2)^n\right)^{\frac{k-l}{k}},\end{aligned}$$

and therefore

$$\left(\int_{\mathbb{B}} (dd^c u)^l \wedge (dd^c|z|^2)^{n-l}\right)^{\frac{1}{l}} \leq C \left(\int_{\mathbb{B}} (dd^c u)^k \wedge (dd^c|z|^2)^{n-k}\right)^{\frac{1}{k}}.$$

Thus,

$$C(0, l, k, n, \mathbb{B}) = \left(\int_{\mathbb{B}} (dd^c|z|^2)^n\right)^{\frac{1}{l} - \frac{1}{k}} = (4\pi)^{\frac{n}{l} - \frac{n}{k}}.$$

This constant is optimal since we have equality in the Poincrè type inequality for the function $|z|^2 - 1$.

Case $p = 1$: As in the case above we shall set $\beta = dd^c(|z|^2 - 1)$, and note that $|z|^2 - 1$ is an exhaustion function for the unit ball $\mathbb{B}$. Let $u \in \mathcal{E}_{p,k}(\mathbb{B})$, $p > 0$. Then by using Hölder's inequality, Theorem 2.5 and integration by parts we get

$$
\begin{aligned}
e_{p,k-1}(u) &= \int_{\mathbb{B}} (-u)^p (dd^c u)^{k-1} \wedge \beta^{n-k+1} \\
&= \int_{\mathbb{B}} (1-|z|^2) dd^c \left(-(-u)^p\right) \wedge (dd^c u)^{k-1} \wedge \beta^{n-k} \\
&\leq p \int_{\mathbb{B}} (1-|z|^2)(-u)^{p-1} (dd^c u)^k \wedge \beta^{n-k} \\
&\leq p \left(\int_{\mathbb{B}} (-u)^p (dd^c u)^k \wedge \beta^{n-k} \right)^{\frac{p-1}{p}} \times \left(\int_{\mathbb{B}} (1-|z|^2)^p (dd^c u)^k \wedge \beta^{n-k} \right)^{\frac{1}{p}} \\
&\leq p\, e_{p,k}(u)^{\frac{p-1}{p}} d(p,k,\mathbb{B})^{\frac{1}{p}} e_{p,k}(u)^{\frac{k}{(p+k)p}} e_{p,k}(|z|^2-1)^{\frac{p}{(p+k)p}} \\
&= p\, d(p,k,\mathbb{B})^{\frac{1}{p}} e_{p,k}(|z|^2-1)^{\frac{1}{p+k}} e_{p,k}(u)^{\frac{p+k-1}{p+k}}.
\end{aligned}
$$

Hence,

$$
e_{p,k-1}(u)^{\frac{1}{p+k-1}} \leq C(p,k,k-1,n,\mathbb{B}) e_{p,k}(u)^{\frac{1}{p+k}}, \tag{4.3}
$$

with

$$
C(p,k,k-1,n,\mathbb{B}) = p^{\frac{1}{p+k-1}} d(p,k,\mathbb{B})^{\frac{1}{p(p+k-1)}} e_{p,k}(|z|^2-1)^{\frac{1}{p+k-1}-\frac{1}{p+k}}.
$$

From (4.3) it now follows

$$
C(p,k,l,n,\mathbb{B}) = C(p,k,k-1,n,\mathbb{B}) \cdot C(p,k-1,k-2,n,\mathbb{B}) \dots C(p,l+1,l,n,\mathbb{B}).
$$

Therefore, for $p = 1$

$$
C(1,k,l,n,\mathbb{B}) = e_{1,k}(|z|^2-1)^{\frac{1}{l}-\frac{1}{k}} = \left(\frac{(4\pi)^n}{n+1} \right)^{\frac{1}{l}-\frac{1}{k}}.
$$

□

Remark. In [42], Trudinger and Wang used the real Hessian quotient operator $\frac{S_k}{S_l}$ to establish the optimal constant in the Poincaré inequality for the real Hessian operator. More precisely, they prove that the optimal constant is attained by the solution u_0 of the equation $\frac{S_k(u_0)}{S_l(u_0)} = 1$. We suspect that this is also the case in the complex setting. With Theorem 4.3 in mind we suspect that the optimal constant is

$$
C(p,k,l,n,\Omega) = (e_{p,k}(u_0))^{\frac{1}{p+l}-\frac{1}{p+k}},
$$

where $p > 0$ and $u_0 \in \mathcal{E}_{p,k}(\Omega)$ is the unique negative k-subharmonic function such that $\mathrm{H}_k(u_0) = \mathrm{H}_l(u_0)$. We refer to [25,38], and reference therein for results concerning such functions in Euclidean spaces as well as on compact manifolds.

5. A Sobolev Type Inequality in m-hyperconvex Domains

Let us first recall the notion of m-capacity. Let $n \geq 2$, $1 \leq m \leq n$. For an arbitrary bounded domain $\Omega \subset \mathbb{C}^n$, and for any $K \Subset \Omega$ define

$$\begin{aligned}\operatorname{cap}_m(K,\Omega) &= \operatorname{cap}_m(K)\\ &:= \sup\left\{\int_K (dd^c u)^m \wedge (dd^c|z|^2)^{n-m} : u \in \mathcal{SH}_m(\Omega), -1 \leq u \leq 0\right\}.\end{aligned}$$

The following lemma was proved by Dinew and Kołodziej [24].

Lemma 5.1. *Let $n \geq 2$, $1 \leq m \leq n$, and let $\Omega \subset \mathbb{C}^n$ be a m-hyperconvex domain. Then for $1 < \alpha < \frac{n}{n-m}$ there exists a constant $C(\alpha) > 0$ such that for any $K \Subset \Omega$,*

$$V_{2n}(K) \leq C(\alpha)\operatorname{cap}_m^\alpha(K).$$

We will also need the following two lemmas.

Lemma 5.2. *Let $n \geq 2$, $1 \leq m \leq n$, $p \geq 0$, and let $\Omega \subset \mathbb{C}^n$ be a m-hyperconvex domain. For $u \in \mathcal{E}_{p,m}(\Omega)$, and any $s > 0$, it holds*

$$\operatorname{cap}_m(\{u < -s\}) \leq 2^{m+p}s^{-m-p}e_{p,m}(u).$$

Proof. By [35,36] we have for any $s, t > 0$

$$\begin{aligned}t^m\operatorname{cap}_m(\{u < -s-t\}) &\leq \int_{\{u<-s\}} (dd^c u)^m \wedge (dd^c|z|^2)^{n-m}\\ &\leq s^{-p}\int_{\{u<-s\}} (-u)^p(dd^c u)^m \wedge (dd^c|z|^2)^{n-m} \leq s^{-p}e_{p,m}(u).\end{aligned}$$

Taking $t = s$ we get

$$\operatorname{cap}_m(\{u < -2s\}) \leq s^{-p-m}e_{p,m}(u).$$

□

Lemma 5.3. *Let $n \geq 2$, $1 \leq m \leq n$, $p \geq 0$, and assume that $\Omega \subset \mathbb{C}^n$ is a m-hyperconvex domain. Then we have that $\mathcal{E}_{p,m}(\Omega) \subset L^q(\Omega)$, for any $0 < q < \frac{n(m+p)}{n-m}$.*

Proof. Assume first that $u \in \mathcal{E}_m^0(\Omega)$, and let $p \geq 0$. Let us define

$$\lambda(s) = V_{2n}(\{u < -s\}).$$

Then by Lemma 5.1, and Lemma 5.2, we get that for $0 < \alpha < \frac{n}{n-m}$

$$\lambda(s) \leq C_1\operatorname{cap}_m^\alpha(\{u < -s\}) \leq C_2 s^{-(m+p)\alpha}e_{p,m}(u)^\alpha,$$

where C_1 and C_2 are constants not depending on u. For $q > 0$ we then have

$$\begin{aligned}\int_\Omega (-u)^q \, dV_{2n} &= q\int_0^\infty s^{q-1}\lambda(s)\, ds = q\int_0^1 s^{q-1}\lambda(s)\, ds + q\int_1^\infty s^{q-1}\lambda(s)\, ds\\ &\le qV_{2n}(\Omega) + C_3 e_{p,m}(u)^\alpha \int_1^\infty s^{q-1-(m+p)\alpha}\, ds\\ &< \infty \Leftrightarrow q < (m+p)\alpha < \frac{n(m+p)}{n-m}, \end{aligned} \tag{5.1}$$

where C_3 is a constant not depending on u. From (5.1) we have $\int_\Omega (-u)^q \, dV_{2n} < \infty$ if, and only if,

$$q < (m+p)\alpha < \frac{n(m+p)}{n-m}.$$

Next, if we take a function $u \in \mathcal{E}_{p,m}(\Omega)$, then there exists a decreasing sequence $u_j \in \mathcal{E}_m^0(\Omega)$ such that $u_j \searrow u$ and $\sup_j e_{p,m}(u) < \infty$. By the first part of the proof there are constants A, B do not depending on u_j such that

$$\|u_j\|_{L^q} \le A + Be_{p,m}(u_j)^\alpha,$$

and by passing to the limit we get

$$\|u\|_{L^q} \le A + B\sup_j e_{p,m}(u_j)^\alpha < \infty.$$

□

Now we can state and prove the Sobolev type inequality in arbitrary m-hyperconvex domains.

Theorem 5.4. *Let $n \ge 2$, $1 \le m \le n$, and $p \ge 0$. Assume that Ω be a bounded m-hyperconvex domain in $\mathbb{C}^n$. There exists a constant $C(p,q,m,n,\Omega) > 0$, depending only on p, q, m, n, and Ω such that for any function $u \in \mathcal{E}_{p,m}(\Omega)$, and for $0 < q < \frac{(m+p)n}{n-m}$, we have*

$$\|u\|_{L^q} \le C(p,q,m,n,\Omega)e_{p,m}(u)^{\frac{1}{m+p}}\,. \tag{5.2}$$

Proof. This follows from Lemma 5.3 and Theorem 3.2. □

We now give examples that shows that the following inequalities are not in general possible:

$$\begin{aligned} e_{p,m}(u)^{\frac{1}{m+p}} &\le C\|u\|_{L^q} \text{ (Example 5.5)}\\ \|u\|_{L^\infty} &\le Ce_{p,m}(u)^{\frac{1}{m+p}} \text{ (Example 5.6)}\\ e_{p,m}(u)^{\frac{1}{n+p}} &\le C\|u\|_{L^\infty} \text{ (Example 5.7)}\,. \end{aligned}$$

Example 5.5. Consider the following functions defined on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$

$$u_j(z) = \frac{1}{j^\alpha}\max\left(1-|z|^{2-\frac{2n}{m}}, 1-j^\beta\right)\,.$$

Then we have

$$u_j(z) = \begin{cases} \frac{1}{j^\alpha}\left(1-|z|^{2-\frac{2n}{m}}\right) & \text{if } j^{\beta\frac{m}{2m-2n}} \leq |z| \leq 1 \\ \frac{1-j^\beta}{j^\alpha} & \text{if } 0 \leq |z| \leq j^{\beta\frac{m}{2m-2n}} \end{cases}.$$

Hence, if $\beta > \alpha\frac{p+m}{p}$, then

$$e_{p,m}(u) = c(n,m)\frac{1}{j^{\alpha(m+p)}}(j^\beta - 1)^p \to \infty, \qquad \text{as } j \to \infty,$$

and

$$c(n,m) = \frac{2\pi^n(\frac{n}{m}-1)^m}{m!(n-m)!}$$

(see [47] for details).

On the other hand, one can check that if $0 < q < \frac{mn}{(n-m)(\beta-\alpha)}$, then

$$\|u_j\|_{L^q}^q \simeq j^{\beta q-\alpha q+\frac{mn}{m-n}} \to 0,$$

as $j \to \infty$. This shows that we can not in general have

$$e_{p,m}(u)^{\frac{1}{m+p}} \leq C\|u\|_{L^q}.$$

□

Example 5.6. Similarly as in Example 5.5 consider the following functions defined on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$

$$u_j(z) = \frac{1}{j^{\frac{p}{m+p}}}\max\left(1-|z|^{2-\frac{2n}{m}}, -j\right).$$

Then we have that

$$\|u_j\|_{L^\infty} = -u_j(0) = j^{\frac{m}{m+p}} \to \infty \qquad \text{as } j \to \infty,$$

and at the same time

$$e_{p,m}(u_j) = c(n,m)j^p\left(\frac{1}{j^{\frac{p}{m+p}}}\right)^{m+p} = c(n,m).$$

Hence, a contradiction is obtained. Thus, we can not in general have

$$\|u\|_{L^\infty} \leq Ce_{p,m}(u)^{\frac{1}{m+p}}.$$

□

Example 5.7. Similarly as before we consider the following functions defined on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$

$$u_j(z) = j\max\left(1-|z|^{2-\frac{2n}{m}}, -\frac{1}{j}\right).$$

Then we have that $\|u_j\|_{L^\infty} = -u_j(0) = 1$, but at the same time

$$e_{p,m}(u_j) = c(n,m)j^{m+p}\left(\frac{1}{j}\right)^p = c(n,m)j^m \to \infty \qquad \text{as } j \to \infty.$$

This shows that we can not in general have

$$e_{p,m}(u)^{\frac{1}{n+p}} \leq C\|u\|_{L^\infty}.$$

□

As an immediate consequence of Theorem 5.4 is that we get that Błocki's integrability conjecture is true for functions in the Cegrell class $\mathcal{E}_m(\Omega)$ (Corollary 5.8). Before stating this result let us recalling the definition of $\mathcal{E}_m(\Omega)$. Let Ω be a bounded m-hyperconvex domain in $\mathbb{C}^n$. We say that $u \in \mathcal{E}_m(\Omega)$ if for any $\omega \Subset \Omega$ there exists $u_\omega \in \mathcal{E}_{0,m}(\Omega)$ such that $u = u_\omega$ on ω.

Corollary 5.8. *Let $n \geq 2$, $1 \leq m \leq n$, and let Ω be a bounded m-hyperconvex domain in $\mathbb{C}^n$. Then $\mathcal{E}_m(\Omega) \subset L^q_{loc}(\Omega)$, for $0 < q < \frac{nm}{n-m}$.*

Acknowledgements

Open access funding provided by Umeå University.

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Per Åhag
Department of Mathematics and Mathematical Statistics
Umeå University
901 87 Umeå
Sweden
e-mail: Per.Ahag@math.umu.se

Rafał Czyż
Institute of Mathematics
Jagiellonian University
Łojasiewicza 6
30-348 Kraków
Poland
e-mail: Rafal.Czyz@im.uj.edu.pl

Received: August 28, 2019.
Accepted: March 17, 2020.